# Return condition for oscillating systems with constrained positive control

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## Problem statement

In this paper we consider the constrained null-controllability problem for the linear system

$$\dot{x} = Ax + bu, \tag{1}$$

without the assumption that the origin is an equilibrium point of the system. In this case trajectories trajectories cannot be held at the point 0 and by controllability we mean being able to reach the origin at any moment of time $T \geq T_0$. In our work we use the concept of the return condition on the interval introduced by V. I. Korobov in the paper [2]. This condition means that for some interval $I$ for any $T \in I$ we can construct a control $u_T(t)$ such that the trajectory starting from the origin can return there in the time $T$.

However this condition is not always easy to check and sometimes we are also interested in constructing the explicit formula for control $u_T(t)$. In our paper we consider the construction of control for the oscillatory system

$$\begin{cases} \dot{x}_{2j-1} = x_{2j}, \\ \dot{x}_{2j} = -j\, x_{2j-1} + u, \end{cases} \quad j = 1, 2, \ldots n, \tag{2}$$

with constraints $u \in [c, 1]$ or $u \in \{c, 1\}, c > 0$.

## Mathematical formulation

Since the solution $x(t)$ of the Cauchy problem

$$\dot{x} = Ax + bu(t), \; x(0) = x_0, \tag{3}$$

has the form

$$x(t) = e^{At}\left(x_0 + \int_0^T e^{-A\tau} bu(\tau) d\tau\right), \tag{4}$$

and $x_0 = x_1 = 0$ we get the condition

$$0 = \int_0^T e^{-At} bu(t) dt. \tag{5}$$

This gives us the trigonometrical momentum problem

$$\begin{cases} \int_0^T \sin jt dt = 0, \\ \int_0^T \cos jt dt = 0, \end{cases} \quad j = 1, 2, \ldots n. \tag{6}$$

Since for $T = 2\pi$ $u(t) = c$ is a solution for any $c$ we are looking the solutions $u_T(t)$ for all $T$ on the interval $I = [2\pi, 2\pi + \alpha], \alpha > 0$ by using the piecewise control

$$u_T(t) = \begin{cases} c, & 0 \leq t \leq T_1, \\ 1, & T_1 \leq t \leq T_2, \\ c, & T_2 \leq t \leq T_3, \\ \ldots \\ 1, & T_{k-1} \leq t \leq T_k, \\ c, & T_k \leq t \leq T, \end{cases} \tag{7}$$

which transforms problem (6) into system of trigonometrical equations

$$\begin{cases} c \sin T_1 + (\sin T_2 - \sin T_1) + \cdots + c(\sin T - \sin T_k) = 0, \\ c \cos T_1 - c + (\cos T_2 - \cos T_1) + \cdots + c(\cos T - \cos T_k) = 0, \\ \ldots, \\ \frac{c}{n} \sin n T_1 + \frac{1}{n}(\sin n T_2 - \sin n T_1) + \cdots + \frac{c}{n}(\sin n T - \sin n T_k) = 0, \\ \frac{c}{n} \cos n T_1 - \frac{c}{n} + \frac{1}{n}(\cos n T_2 - \cos n T_1) + \cdots + \frac{c}{n}(\cos n T - \cos n T_k) = 0. \end{cases} \tag{8}$$

## Solution with $2n$ switching points

For $c = \frac{1}{2}$ it is possible to write the general explicit solution with $2n$ switching points. For $T = T + a, 0 < a < \alpha$ It has the following form:

$$u_n(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq \frac{2\pi}{n+1}, \\ 1, & \frac{2\pi}{n+1} \leq t \leq \frac{2\pi}{n+1} + a, \\ \frac{1}{2}, & \frac{2\pi}{n+1} + a \leq t \leq 2\frac{2\pi}{n+1}, \\ 1, & 2\frac{2\pi}{n+1} \leq t \leq 2\frac{2\pi}{n+1} + a, \\ \ldots \\ 1, & n\frac{2\pi}{n+1} \leq t \leq n\frac{2\pi}{n+1} + a, \\ \frac{1}{2}, & n\frac{2\pi}{n+1} + a \leq t \leq 2\pi + a. \end{cases} \tag{9}$$

The graph control for $n = 6$, $c = \frac{1}{2}$, $a = 0.1$ is shown in Figure 1, the individual trajectories are shown in Figure 2. In Figures 3 and 4 the phase trajectories for two first and two last coordinates are shown.

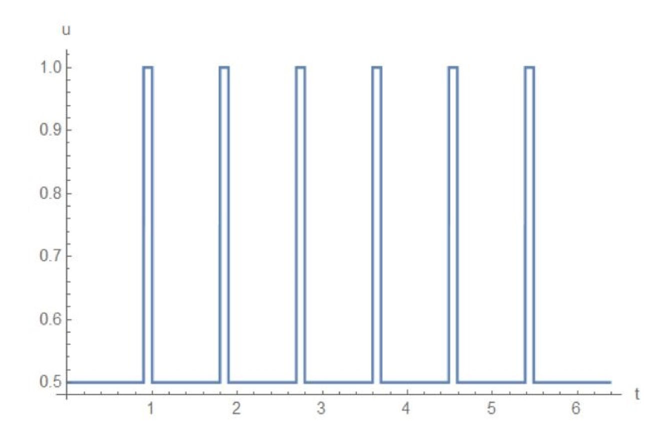

Figure 1. Graph of control

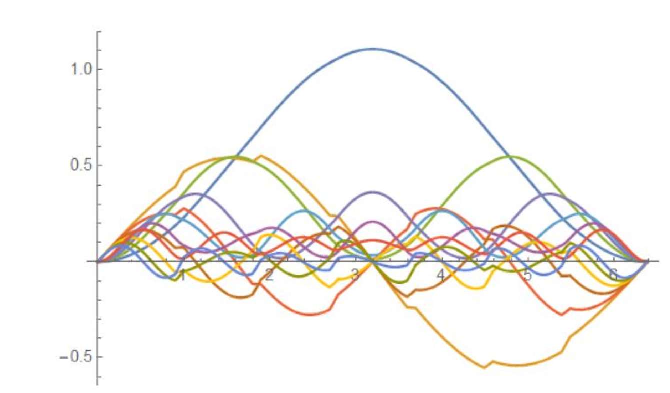

Figure 2. Individual trajectories

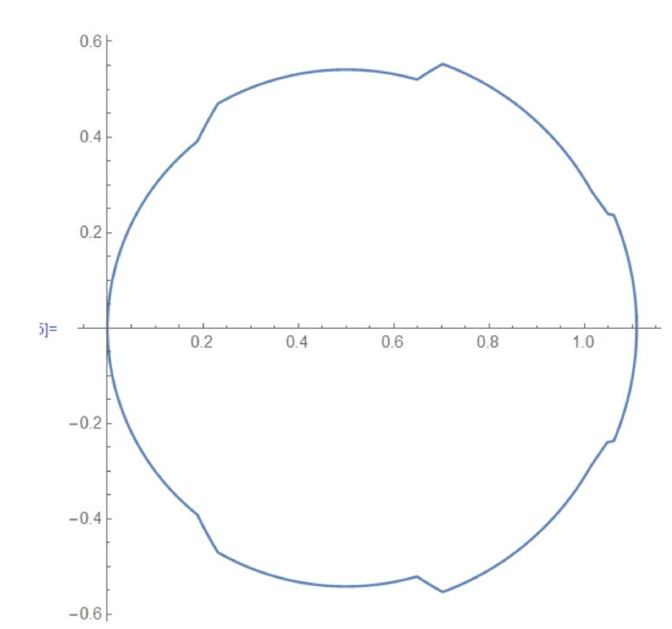

Figure 3. Phase trajectory for $x_1, x_2$

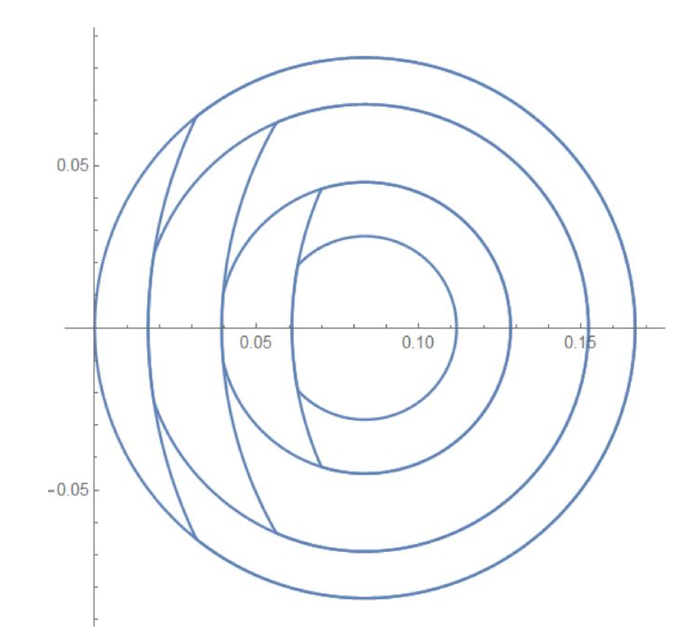

Figure 4. Phase trajectory for $x_{11}, x_{12}$

For $c \neq \frac{1}{2}$ it is harder to obtain general solution. For the case $n = 1$ we were able to obtain it in explicit form:

$$T_1 = \arctan\left(\frac{\sin\left(\frac{a}{2}\right)\left(\sqrt{2\left(\cos(a) + 2\left(\frac{1}{c}\right)^2 - \frac{4}{c} + 1\right)} - 2\cos\left(\frac{a}{2}\right)\right)}{\cos\left(\frac{a}{2}\right)\sqrt{2\left(\cos(a) + 2\left(\frac{1}{c}\right)^2 - \frac{4}{c} + 1\right)} - \cos(a) + 1}\right) + \pi,$$
$$T_2 = \arctan\left(\frac{\sin\left(\frac{a}{2}\right)\left(\sqrt{2\left(\cos(a) + 2\left(\frac{1}{c}\right)^2 - \frac{4}{c} + 1\right)} + 2\cos\left(\frac{a}{2}\right)\right)}{\cos\left(\frac{a}{2}\right)\sqrt{2\left(\cos(a) + 2\left(\frac{1}{c}\right)^2 - \frac{4}{c} + 1\right)} + \cos(a) - 1}\right) + \pi. \tag{10}$$

## Solution with 2 switching points

Using the symmetry of the problem for $c = \frac{1}{2}$ we can reduce the number of switching points to only 2 for any size $n$. For this we write the momentum problem in exponential form

$$\int_0^T u(t) e^{kit}\, dt = 0, \;\; k = 1, 2, ..., n. \tag{11}$$

and consider control

$$u(t) = \begin{cases} c, & 0 \leq t \leq T_1, \\ 1, & T_1 \leq t \leq T_2, \\ c, & T_2 \leq T. \end{cases} \tag{12}$$

with $T - T_2 = T_1 - 0$. By substituting $e^{T_1} = x, e^T = s \implies e^{T_2} = \frac{s}{x}$ we get the system of equations for $x$ and $s$:

$$\begin{cases} -c + (c-1)x + (1-c)\frac{s}{x} + cs = 0, \\ -c + (c-1)x^2 + (1-c)\frac{s^2}{x^2} + cs^2 = 0, \\ \ldots \\ -c + (c-1)x^n + (1-c)\frac{s^n}{x^n} + cs^n = 0, \end{cases} \tag{13}$$

It always has a solution $x = s$, so we can choose $T = 2\pi + T_1$, $T_2 = 2\pi$. On Figures 5 and 6 the trajectories for individual and the pairs of coordinates are shown.

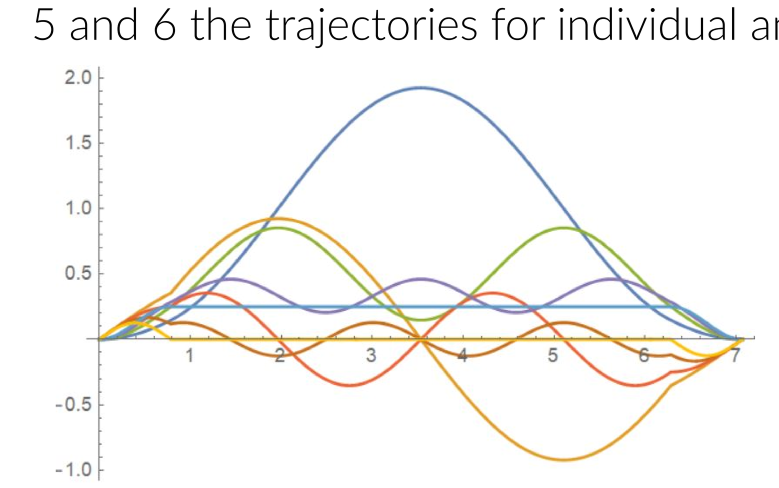

Figure 5. Individual trajectories for $n = 4$

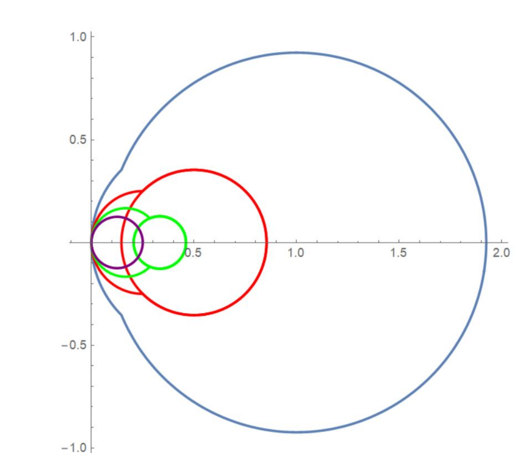

Figure 6. Pairs trajectories for $n = 4$

It also should be noted that this solution does not depend on problem size $n$. Instead of control (12) we can also choose

$$u(t) = \begin{cases} c, & 0 \leq t \leq T_1, \\ 1, & T_1 \leq t \leq T_2, \\ 1 - c, & T_2 \leq T. \end{cases} \tag{14}$$

## Generalization

Since the system (8) depends only on exponent of matrix $A$ and vector $b$, the control (12) is true for any $n$ and for any set of rational numbers we can find a common multiple divisible by $2\pi$ the following theorem holds

**Theorem** *For the system*

$$\dot{x} = Ax + bu, \; c \leq u \leq 1, \, c \leq \frac{1}{2}. \tag{15}$$

*with matrix $A$ of size $2n \times 2n$ and simple imaginary eigenvalues $\pm i\nu_k$, $k = 1, \ldots, n$ and such that* rank $(b, Ab, \ldots, A^{2n-1}b) = 2n$, *the return condition is satisfied if $\nu_k$ are rational numbers.*

## References

[1] Bianchini, R. M.,Local Controllability, Rest States, and Cyclic Points, SIAM Journal on Control and Optimization, Vol. 21, pp. 714–720, 1983.

[2] Korobov, V.I. Geometric Criterion for Controllability under Arbitrary Constraints on the Control. J Optim Theory Appl 134, 161–176 (2007). https://doi.org/10.1007/s10957-007-9212-2.

[3] Margheri, A. On the 0-local controllability of a linear control system. J Optim Theory Appl 66, 61–69 (1990). https://doi.org/10.1007/BF00940533.

[4] A. M. Zverkin, V. N. Rozova, "Reciprocal controls and their applications", Differ. Uravn., 23:2 (1987), 228–236.